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An improved time marching algorithm for GWC shallow water models

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Abstract

Finite element solutions of the shallow water wave equations have found increasing use by researchers and practitioners in the modeling of oceans and coastal areas. Wave equation models successfully eliminate spurious oscillation modes without resorting to artificial or numerical damping. Typically, wave equation models integrate the continuity equation with a three-time-level scheme centered at k and the momentum equation with a two-time-level scheme centered at k+1/2; nonlinear terms are evaluated explicitly. This allows for a computationally-efficient sequential solution procedure. However in highly nonlinear applications, the algorithm becomes unstable for high Courant numbers. In this work, we examine a predictor-corrector algorithm to improve the stability constraint. Two advantages of the predictor-corrector scheme over the alternative of simultaneous integration of the full nonlinear equations are: 1) they can be easily implemented within the framework of existing codes; and 2) they minimize the size of the matrices that must be stored and inverted. Results from an exhaustive series of one-dimensional experiments show that, depending on the bathymetry, grid resolution, and iteration procedure, we can see over a ten-fold increase in the size of the stable time step. Implementation of the most promising algorithm into the 2D/3D circulation model, Adcirc, is in progress.

Background

Shallow water equations are based on the depth-averaged equations of motion subject to the assumption of a hydrostatic pressure distribution; they describe the propagation of long water waves in oceans, estuaries, and impoundments. Early finite element algorithms were plagued by spurious oscillations superimposed on the true solution. The oscillations were damped with either artificially high viscous coefficients or with a dissipative time marching scheme[1]. Lynch and Gray[2] introduced the wave continuity equation in 1979 as a means to successfully suppress the noise without damping the true solution. Since its inception, the wave continuity algorithm has been modified in a number of substantial ways[3-7] so that the resulting code (referred to herein as Adcirc[8]) can accurately model the three-dimensional hydrodynamic behavior of coastal and oceanic areas[9-11].

While the Adcirc model has proven to be a efficient and accurate simulator, in some highly nonlinear applications, the sequential solution procedure imposes a severe Courant number restriction on the allowable time step. For example, with the Western North Atlantic model[9], the Courant number (as determined by the linear wave celerity) must be on the order of 0.5 in order for the simulation to remain stable. With highly advective flows, such as flow around barrier islands and constricted inlets, the stability constraint is even more extreme. One obvious solution would be to implement a fully-implicit, simultaneous solution of the nonlinear continuity and momentum equations. However, this strategy would be inconsistent with the framework of the existing Adcirc code, and it would require storing and inverting larger matrices. Herein, we propose an alternative iterative technique (predictor-corrector) that conforms to Adcirc's sequential solution algorithm. It is tested extensively with a one-dimensional version of the code applied to a variety of problems.

Conservation equations

The full equations can be found in a variety of sources[8,12]; in the interest of brevity, only the operator notation is given here. Let L represent the primitive continuity equation, $\mathbf{M}$ the non-conservative form of the momentum equation (NCM), and $\mathbf{M}^{\mathrm{c}}$ the conservative form of the momentum equation. The generalized wave continuity equation (GWC) is obtained from

$$W^G \equiv \frac{\partial L}{\partial t} + GL - \nabla \bullet \mathbf{M}^{\mathrm{c}} = 0 \tag{1}$$

where G is a numerical parameter. The wave continuity equation, as it originally appeared in Lynch and Gray[2], is obtained by setting $G = \tau$ where τ represents the bottom friction. Note that the primitive continuity equation can be viewed as a limiting form of the GWC equation by letting $G \to \infty$.

Original solution algorithm

Semi-discrete equations are obtained by interpolating W^G and $\mathbf{M}$ with C^0 linear finite elements (triangles). Implicit time discretization of W^G uses a three-time-level approximation centered at k. Time discretization of $\mathbf{M}$ uses a lumped two-time-level approximation centered at $k + 1/2$. Equations are linearized by formulating the advective terms explicitly. Exact quadrature rules are used. A time-splitting solution procedure is adopted wherein W^G is first solved for nodal elevations and then $\mathbf{M}$ is solved for the velocity field. The resulting discrete equations can be found in Luettich et al.[8]

Alternative time marching algorithm

The algorithm

The alternative algorithm uses the original time stepping procedure to predict the elevation and velocity field at the new time level. These predicted quantities are then used to update some or all of the nonlinear terms in the right-hand side load vector (recall that the nonlinear terms are evaluated explicitly), after which a corrected elevation and velocity field are obtained. The predictor-corrector loop can be repeated until convergence.

Numerical experiments

To evaluate the relative effect of each of the nonlinear terms, an exhaustive set of one-dimensional experiments was conducted on a model problem. Table 1 gives the experimental matrix. Conditions for the model problem are: 1D channel, 50 km long reach, 5 meters deep, constant bottom friction of 10^{-4} sec^{-1}, a G/τ ratio of 10 (constant), eddy viscosity is zero, 50 equally-sized elements, and forced with a 1 meter M_2 tide at the ocean boundary and no normal flux at the opposite land boundary. The model problem was chosen because its shallow bathymetry and large amplitude forcing creates significant nonlinearities; it has been a very useful test problem in our previous analyses. While we recognize that positive 1D results are no guarantee for success in 2D, our experience has shown that if it *does not* work in 1D, it certainly will not work in 2D applications.

Table 1. Numerical stability experiments for the model problem.

Group #	Nonlinear Term Subject to Iterative Improvement[1]	Parameter(s) Varied[2]	Maximum Courant #[3]	% Change from Base[4]
1	NCM advective	*form* of the term (cons./non-cons.); no iterative improvement	0.95	0.0%
2	GWC advective	# iterations; center at k or k+1/2	1.02	7.4%
3	NCM advective	# iterations; center at k or k+1/2	1.12	17.9%
4	GWC finite amplitude	# iterations; center at k or k+1/2	0.95	0.0%
5	GWC flux times G	# iterations; center at k or k+1/2	1.51	55.8%
6	GWC flux times τ	# iterations; center at k or k+1/2	0.95	0.0%
7	Permutations of Groups 2 through 6	center at k or k+1/2	2.14	121%[5]

1. NCM advective refers to the $\mathbf{v} \bullet \nabla \mathbf{v}$ term in the NCM equation.
   GWC advective refers to the $\nabla \bullet (H\mathbf{v}\mathbf{v})$ term in the GWC equation.
   GWC finite amplitude refers to the $gH\nabla\zeta$ term in the GWC equation.
   GWC flux times G refers to the $G\nabla \bullet (H\mathbf{v})$ term in the GWC equation.
   GWC flux times τ refers to the $\tau\nabla \bullet (H\mathbf{v})$ term in the GWC equation.
2. Centered at k means the term was weighted equally between k-1, k, k+1.
   Centered at k+1/2 means the term was weighted equally between k, k+1.
3. Courant # was calculated as $c\Delta t/\Delta x$ where $c = \sqrt{gh}$ and h is bathymetry.
4. For a base run, the code is run with both advective terms in conservative form and no iterative improvement.
5. Maximum realized by iterating on the GWC flux times G term centered at k and the NCM advective term centered at k+1/2, ie, a combination of Group 3 and 5.

Past work has shown that accuracy, and algorithm behavior in general, is very sensitive to the ratio of G/τ. For the set-up in Group 7 that produced the 121% change (Table 1, footnote 5), we explored its sensitivity to the value of G by varying it over several orders of magnitude.

Another complete set of 1D experiments was conducted in order to assess the effects of variable grid spacing and variable bathymetry on the stability. The domains are shown in Figure 1 and Figure 2; the former represents a quadratic-like change in bathymetry with varying rates of rise, while the latter represents a simplified cross-section of the Western North Atlantic extending from the East coast of the US to the deep ocean. For each, different levels of discretization were used (both constant and variable node spacing), ranging from a minimum M_2 resolution of 30 nodes per wavelength to over 1000. In addition to evaluating the individual effects of each nonlinear term with central time weighting, a la Groups 2-6 in Table 1, we experimented with time weights that were not centered in an effort to find the maximum stable time step.

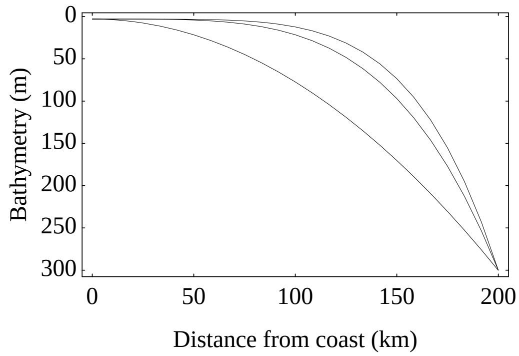

Figure 1. Bathymetry for quadratic test problems.

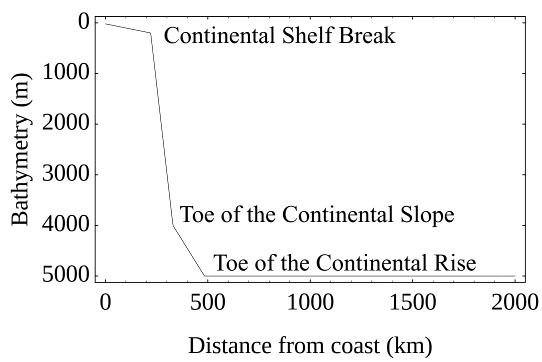

Figure 2. Bathymetry for Western North Atlantic test problems.

Results

Stability was defined as the maximum allowable time step, to the nearest 5 seconds, that could be used without causing overflow errors. Table 1 summarizes key results for the first set of experiments. In the table, percent change refers to the increase in the time step as compared to the original solution algorithm. Note that since each iteration requires another matrix inversion, only iterations that show more than a 100% change per iteration are cost-effective. (This is a conservative estimate in that it assumes the entire load vector is re-evaluated each iteration, while in reality, only the nonlinear term(s) are affected.)

As can be seen in the table, the form of the advective terms, the finite amplitude terms, and the flux times τ term have absolutely no influence on the stability of the model problem. (However, it should be noted that they *do* have a significant effect on the overall accuracy of the code.) As Group

2 and 3 results show, iterative improvement using the advective terms offers some gain in stability, but the most influence of a *single* term is the GWC flux times G term (Group 5). For this model problem, we were able to overcome the "break-even" point, ie, over 100% improvement, for the Group 7 simulation that iterated upon the NCM advective term in conjunction with the GWC flux times G term. It is interesting to note that the combined effect is more than the sum of the improvements shown by the individual parts. In an effort to minimize computational costs, we did try iterating every other time step. Even considering the reduced cost, the gains were not as significant so this alternative was not explored further.

Concerning sensitivity to the numerical G parameter, we found that, for the conditions of the model problem (the constant depth channel), stability is maximum when the G/τ ratio is on the order of 2 to 50, as shown in Figure 3. In a previous study[12], we have shown that mass balance errors and errors in the generation of nonlinear constituents are minimized when the ratio is on the order of 1 to 10. Thus, it appears that the increase in stability can be made without sacrificing accuracy and without using so high of a G value that spurious oscillations are introduced.

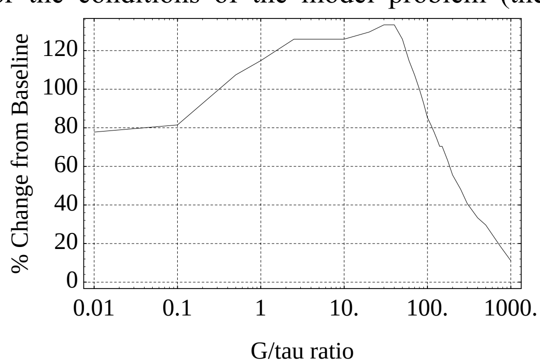

Figure 3. Sensitivity of stability to the G/τ ratio for conditions of the model problem.

Full results from the second set of experiments on the other domains and other grid discretizations are not shown because of space limitations. In general, they confirm the results shown in Table 1, viz, when any one single nonlinear term was iterated upon, the maximum increase in the allowable time step was seen when that term was the GWC flux times G term weighted equally at time levels k-1, k, and k+1. Furthermore, when iterated upon in conjunction with the other nonlinear terms using optimum time weights, the percent change ranged from 200% for the quadratic-like bathymetry (Figure 1) with variable Δx, to over 1000% for both the quadratic and East coast bathymetry (Figure 2) with constant Δx.

Conclusions

Results from the 1D experiments indicate the following:

- The limiting Courant number without iterative improvement is 1.0.
- With iterative improvement, more than a 100% gain in the maximum stable time step can be realized for all test problems herein. For some of the simulations, we realized over a ten-fold increase.
- The nonlinear term that most influences stability is the $G\nabla\bullet(H\mathbf{v})$ term in the GWC equation, while the second most influential is the advective term in the NCM equation.
- For conditions of the model problem, the stability gained from the iterative improvement on the above terms is maximum when the G/τ ratio is greater than 1, but less than 50. However, we recommend an upper bound of 10 be used in order to avoid spurious oscillations.
- For nearly all experiments, using two iterations results in only marginal gains in stability, while going from two to three iterations gave no improvement for any of the experiments. Thus, it appears the solution has converged to near machine accuracy after a single iterative improvement loop.
- For all nonlinear terms in the GWC equation, weighting terms equally between time levels *k*-1, *k*, and *k*+1 resulted in the most consistent increase in the allowable time step. For nonlinear terms in the NCM equation, equal weighting between time levels *k* and *k*+1 gave the most consistent improvements. This result is in accord with the time weights used for other terms in the respective equations.
- In all problems, we found a higher percent increase in the allowable Δt, and a higher absolute Δt, when the node spacing was constant.
- Maximum improvement and the optimum set of time weights is grid and problem dependent. In lieu of extensive trial and error experiments, we recommend that two nonlinear terms be updated with a single predictor-corrector loop: 1) the GWC flux times *G* term with equal time weights between *k*-1, *k*, and *k*+1; 2) the NCM advective term with equal time weights between *k* and *k*+1.

Future work

While the results from the one-dimensional experiments are encouraging, additional testing, analyses, and sensitivity studies are needed, including the effects of boundary conditions and other grid spacing algorithms besides the $\lambda/\Delta x$ criterion. However, given the positive results herein, we are proceeding to test the predictor-corrector time marching algorithm in the 2D/3D Adcirc code.

References

1 Wang, J.D. and Connor, J.J., Mathematical Modeling of Near Coastal Circulation, MIT Parsons Lab. Report No. 200, Cambridge, MA, 1975.

2 Lynch, D.R. and Gray, W.G., A Wave Eqn Model for Finite Element Tidal Computations, *Comp. and Fluids*, 7(3), pp. 207-228, 1979.

3 Kinnmark, I.P.E., *The Shallow Water Wave Equations: Formulation, Analysis, and Application*, Lecture Notes in Engineering, No. 15, Springer-Verlag, 1986.

4 Kolar, R.L. and Gray, W.G., Shallow Water Modeling in Small Water Bodies, *Computational Methods in Surface Hydrology*, eds. G. Gambolati et al., Computational Mechanics Publications/Springer-Verlag, pp. 149-155, 1990.

5 Naimie, C.E. and Lynch, D.R., Applications of Nonlinear Three-Dimensional Shallow Water Equations to a Coastal Ocean, *Computational Methods in Water Resources IX, Vol. 2: Mathematical Modeling in Water Resources*, eds. Russell et al., Computational Mechanics/ Elsevier, Southampton/London, pp. 589-608, 1992.

6 Kolar, R.L., Gray, W.G. and Westerink, J.J., Boundary Conditions in Shallow Water Models - An Alternative Implementation for Finite Element Codes, *Int'l J. for Num. M. in Fluids*, 22, pp. 603-618, 1996.

7 Chippada, S., Dawson, C.N., Wheeler, M.F. and Martinez, M.L., Parallel Computing of Finite Element Surface Water Flows, *CMWR XI. Volume 2: Computational Methods in Surface Flow and Transport Problems*, eds. Aldama et al., pp. 63-70, 1996.

8 Luettich, R.A., Westerink, J.J., and Scheffner, N.W., *ADCIRC: An Advanced Three-Dimensional Circulation Model for Shelves, Coasts, and Estuaries*, Dept. of the Army, U.S. Army Corps of Engineers, Washington, D.C., 1991.

9 Kolar, R.L., Westerink, J.J., Gray, W.G. and Luettich, R.A., Shallow Water Modeling in Spherical Coordinates: Equation Formulation, Numerical Implementation, and Application, *J. of Hydraulic Research*, 32(1), pp. 3-24, 1994.

10 Westerink, J.J., Luettich, R.A., Baptista, A.M., Scheffner, N.W., and Farrar, P., Tide and Storm Surge Predictions Using a Finite Element Model, *Journal of Hydraulic Engineering*, 118, pp. 1373-1390, 1992.

11 Gray, W.G., A Finite Element Study of Tidal Flow Data for the North Sea and English Channel, *Adv. in Water Res.*, 12(3), pp. 143-154, 1989.

12 Kolar, R.L., Westerink, J.J., Cantekin, M.E. and Blain, C.A., Aspects of Nonlinear Simulations Using Shallow Water Models Based on the

Wave Continuity Eqn., *Comp. and Fluids*, 23(3), pp. 523-538, 1994.

Key Words

time marching, stability, shallow water models, wave continuity equation, finite elements