Appeared in: *Computational Methods in Water Resources X, Volume 2,* Peters et al. (eds.), Kluwer Academic Publishers, pp. 1081-1088, 1994.

NORMAL FLOW BOUNDARY CONDITIONS IN SHALLOW WATER MODELS - INFLUENCE ON MASS CONSERVATION AND ACCURACY

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ABSTRACT

Finite element solution of the shallow water wave equations has found increasing use by researchers and practitioners in the modeling of oceans and coastal areas. Wave equation models, most of which use equal-order, C^0 interpolants for both the velocity and the surface elevation, successfully eliminate spurious oscillation modes without resorting to artificial or numerical damping. An important question for both primitive equation and wave equation models is the interpretation of boundary conditions. Analysis of the characteristics of the governing equations shows that a single condition at each boundary is sufficient. Yet there is not a consensus in the literature as to what that boundary condition must be or how it should be implemented in a finite element code. Traditionally (partly because of limited data) surface elevation is specified at open ocean boundaries while the normal flux is specified as zero at land boundaries. In most finite element wave equation models, both of these boundary conditions are implemented as essential conditions. Our recent work focuses on alternate ways to numerically implement normal flow boundary conditions with an eye toward improving the mass-conserving properties of wave equation models. In particular, we have found that treating normal fluxes as natural conditions with the flux interpreted as external to the computational domain results in a mass conservative scheme for all parameter values. Use of generalized functions in the finite element formulation shows this is a natural interpretation. A series of two-dimensional experiments demonstrates that this interpretation also improves the accuracy of primitive equation models by eliminating some of the spurious oscillation modes.

BACKGROUND

Shallow water equations are obtained by vertically averaging the microscopic mass and momentum balances over the depth of the water column. Early finite element solutions of the shallow water equations were often plagued by spurious oscillations. Various methods were introduced to eliminate the oscillations but all included some type of artificial damping. Lynch and Gray [1] and Gray [2] present the wave continuity equation as a means to successfully suppress spurious oscillations without resorting to numerical or artificial damping of the solution. Since the inception of the wave continuity formulation in 1979, the original algorithm has been modified in a number of substantial ways: a numerical parameter was introduced to provide a more general means of weighting the primitive continuity equations [3]; viscous dissipation terms were incorporated [4, 5]; and three-dimensional simulations were realized by resolving the velocity profile in the vertical [6, 7]. The resulting algorithm has been extensively tested using analytical solutions and field data and is currently being used to model the hydrodynamic behavior of coastal and oceanic areas [8-10].

In the course of some of the applications, it was discovered that when nonlinear components of the solution are significant, the wave continuity equation in its original form does not conserve mass. Two methods of mitigating the errors are presented in Kolar et al. [11]. In the first, it is shown that if G, the numerical parameter in the generalized wave continuity equation, is increased so that its value is one or two orders of magnitude larger than the bottom friction, then mass conservation is greatly improved. However, an upper bound on G exists above which the solution becomes too primitive and spurious oscillations appear in the solution. Dispersion analysis can be used as a tool to *a priori* predict the maximum value of G. The second mitigation technique reformulates the convective term in the generalized wave continuity equation so that a consistency exists between the momentum and continuity equations (e.g., both equations cast the advective terms in non-conservative form). If both mitigating measures are used in conjunction, then global mass balance errors are eliminated while errors in local regions (even individual elements) are virtually nonexistent except for regions near the open boundaries. One and two-dimensional applications demonstrate the effectiveness of the procedure. However, the persistence of errors near the open boundaries and absence of errors near land boundaries, has led to this study of the effect of boundary conditions on mass conservation and solution accuracy.

CONSERVATION EQUATIONS

Primitive forms of the balance laws are obtained by vertical averaging of the microscopic balance laws. Using operator notation, we present the primitive form of conservation of mass (continuity equation) as

$$L \equiv \frac{\partial \zeta}{\partial t} + \nabla \bullet (H\mathbf{v}) = 0 \tag{1}$$

The conservative form of conservation of momentum is given by

$$\mathbf{M}^c \equiv \frac{\partial (H\mathbf{v})}{\partial t} + \nabla \bullet (H\mathbf{v}\mathbf{v}) + \tau H\mathbf{v} + H\mathbf{f} \times \mathbf{v} + gH\nabla\zeta - \mathbf{A} - \frac{1}{\rho}\nabla \bullet (H\mathbf{T}) = 0 \tag{2}$$

and the non-conservative form of conservation of momentum is given by

$$\mathbf{M} \equiv \frac{1}{H}(\mathbf{M}^c - \mathbf{v}L) = 0 \tag{3a}$$

Substituting (1) and (2) into (3a) gives

$$\mathbf{M} \equiv \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \bullet \nabla \mathbf{v} + \tau \mathbf{v} + \mathbf{f} \times \mathbf{v} + g\nabla\zeta - \frac{\mathbf{A}}{H} - \frac{1}{\rho H}\nabla \bullet (H\mathbf{T}) = 0 \tag{3b}$$

In operator form, the generalized wave continuity (GWC) equation is

$$W^G \equiv \frac{\partial L}{\partial t} + GL - \nabla \bullet \mathbf{M}^c = 0 \tag{4a}$$

Substituting (1) and (2) into (4a) gives

$$W^G \equiv \frac{\partial^2 \zeta}{\partial t^2} + G\frac{\partial \zeta}{\partial t} + (G - \tau)\nabla \bullet (H\mathbf{v}) - \nabla \bullet \Big[\nabla \bullet (H\mathbf{v}\mathbf{v}) + H\mathbf{f} \times \mathbf{v} + gH\nabla\zeta$$

$$- \mathbf{A} - \frac{1}{\rho}\nabla \bullet (H\mathbf{T})\Big] - H\mathbf{v} \bullet \nabla\tau = 0 \qquad \textbf{(4b)}$$

The wave continuity equation, as it originally appeared in Lynch and Gray [1], is obtained by setting $G = \tau$. Note that the primitive continuity equation can be viewed as a limiting form of the generalized wave continuity equation by letting $G \to \infty$.

DISCRETIZATION

Equations (1), (2), (3b), and (4b) are discretized in space using a standard Galerkin finite element approximation with linear elements. Implicit time discretization of L and W^G uses a three time level approximation centered at k. Time discretization for $\mathbf{M}$ and $\mathbf{M}^c$ uses a lumped two time level approximation centered at $k + 1/2$; the discrete equations are linearized by formulating the advective terms explicitly. Exact quadrature rules are employed. The resulting discretized equations can be found in [6]. A sequential solution procedure is adopted where the continuity equation ((1) or (4b)) is used to solve for elevations and the momentum equation ((2) or (3b)) is used to solve for the velocity field.

BOUNDARY CONDITIONS

The governing conservation equations represent a coupled hyperbolic system of partial differential equations that describe the propagation of long water waves in shallow water. As such, characteristic theory is an appropriate tool to study proper specification of boundary conditions. In particular, for the primitive conservation equations, it has been shown that one condition on each physical boundary is required (in addition to initial conditions for the "time boundary"). Drolet and Gray [12] extend the analysis of characteristic planes to the wave continuity equation and determined that a single boundary condition is still sufficient.

Mathematically, the conditions are specified as one of three types: Dirichlet (Type I) in which the value of the dependent variable (elevation or flux) is specified, Neumann (Type II) in which the value of the flux is specified, and Robin (Type III) which is a linear combination of the first two. In shallow water modeling, these types describe the physical situations of known-elevation, known-flux, or stage-discharge relations, respectively (the latter are often referred to as radiation boundary conditions). This article focuses on the first two types of conditions. In finite element vernacular, a Dirichlet condition means that the value of the dependent variable is known on the boundary; it is referred to as an essential condition. A specified-flux condition enters the right hand side vector in the set of discrete equations and is referred to as a natural boundary condition.

For a single partial differential equation, such as Laplace's equation or the diffusion equation, specified boundary conditions fall neatly into one of the above categories and implementation is unambiguous. Unfortunately, such is not the case for the coupled hyperbolic system at hand, for what may serve as an essential condition for the momentum equation may equally be interpreted as a natural condition for the continuity equation. This ambiguity has led to an inconsistent treatment of boundary conditions in the literature; to date, no consensus on the "best" way to implement the conditions exists. Complicating the matter is the fact that data often dictates what information is available at the boundary. For example, elevation data, either from global tidal models or from field measurements, is more reliable and more prevalent than is velocity data. The end result is that the researcher is often faced with the task of choosing one interpretation over the other, which, frequently, is tantamount to choosing which boundary equation to discard.

Lynch [13] seems to be the first to study the effect of boundary conditions on mass conservation in the context of the wave continuity algorithm. Using the well-known properties of linear basis functions that the sum of the functions over all elements is equal to one and the sum of the gradient of the basis functions over all elements is zero, he demonstrates that all terms of the continuity equation must be retained in order to maintain global conservation of mass, regardless of the nature of the boundary data. He refers to this interpretation of the boundary conditions as mass conservative boundary conditions. However, several open issues remain. For example, it is not clear how momentum conservation is affected by this interpretation - an equally-important consideration.

Accordingly, we undertook an extensive series of one-dimensional experiments to study various means of implementing mass conservative boundary conditions. A shallow one-dimensional channel was used for the model problem so that significant nonlinear components are generated. Conditions for the problem are:

channel coordinates	$0 \le x \le 50$ km
channel depth	5 m
eddy viscosity ε	0.093 m^2/sec
Δt	25 sec
Δx	2 km
boundary conditions	$\zeta(0, t) = 1.0 \sin[2\pi t/12.42 \text{ hrs}]$ m
	$u(50, t) = 0.0$ m/sec
initial conditions	cold start: $\zeta(x, 0) = u(x, 0) = 0.0$
bottom friction τ	0.0001 sec^{-1} (constant)

The x-axis is defined positive to the right. The boundary conditions describe a channel with a land boundary at $x = 50$ km being forced by an M_2 tide with 1 meter amplitude at $x = 0$ km.

Mass conservation was checked globally and locally by comparing mass accumulation with cumulative net flux for the region of interest. Details of the algorithm are presented in [11]. For model comparison, a fine grid solution using primitive balance laws, which is mass conservative, was taken as the true solution. Note that for one-dimensional simulations with constant bathymetry, spurious oscillations do not plague the solution so that a primitive solution is satisfactory.

Conventional formulation of the boundary conditions implements all boundary information through the use of essential boundary conditions. That is, specified elevation results in a reduced matrix for the continuity equation and specified velocity results in a reduced matrix for the momentum solution with the corresponding finite element equation discarded. This interpretation leads to gross mass balance errors as demonstrated in Figure 1. For perfect mass balance, the two curves should overlay one another. (The mitigating procedures discussed earlier are not implemented here so as to isolate the effect of boundary conditions.)

Results of the numerical experiments show that the key to using mass conservative boundary conditions with C^0 basis functions is interpretation. Specifically, state variables that appear in boundary integrals must be interpreted as external to the domain. To fix ideas, consider the finite element formulation of the primitive equations((1) and (3b)) for the one-dimensional model problem where Green's Theorem has been applied to the flux term in (1). The weak form is given by

$$\langle \frac{\partial \zeta}{\partial t}, \varphi_i \rangle - \langle Hu, \frac{\partial \varphi_i}{\partial x} \rangle + Q\Big|_{x_{0-}}^{x_{50+}} = 0 \tag{5}$$

Proper interpretation requires that the boundary flux term, $Q = Hu$, be viewed as external to the domain, i.e., an unknown quantity. Hence the $\pm$ designation on the limits of integration. In this way, the number of equations plus boundary conditions is equal to the number of unknowns so that all information is used; no equations are discarded. Thus, at the left boundary where ζ is specified, elevation is known so that (5) is solved for external flux. The momentum equation is then used to solve for velocity at the left boundary, u_{0+}. At the right boundary, the flux is known so it enters the finite element formulation given in (5) naturally, and the equation is solved for the unknown elevation. The momentum equation is then used to solve for the unknown velocity at the right boundary, u_{50-}. Note that such an interpretation allows for a discontinuity in velocity at the boundary.

When this external flux interpretation was applied to the GWC algorithm, simulation of the model problem results in no global mass balance error as shown in Figure 2. A large number of additional one-dimensional experiments, some looking at mixed formulations, were conducted to test the hypothesis. In one particularly interesting experiment, Green's Theorem was applied to the finite amplitude term, $g\nabla\zeta$, in the momentum equation. Consistent with the external flux interpretation, this boundary term had to be interpreted as external to the domain in order to realize stable, accurate, mass conservative results. Alternative interpretations (for example, treating specified flux as both essential in the momentum balance and natural in the continuity equation) that were tested led to unstable algorithms.

Additional evidence to support this interpretation of boundary terms comes from two sources. First, Westerink et al. [14] conducted a number of two-dimensional numerical experiments

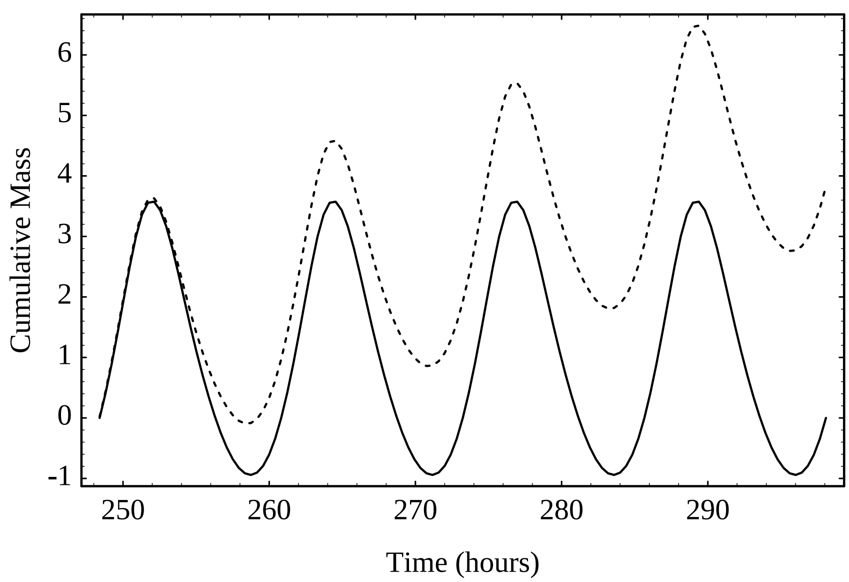

Figure 1 Global mass balance check of the GWC formulation ($G = \tau$) for the model problem, conventional interpretation of boundary conditions.

on a fictitious harbor with a known analytical solution. Both primitive and GWC algorithms were tested. One of the conclusions from the paper is that when known boundary fluxes are treated as external to the domain and implemented as natural conditions in the continuity equation, then the accuracy of the solution improves. In particular, $2\Delta x$ oscillations are damped. Experimental results were supported with dispersion analyses of both interior and boundary equations. Second, Gray and Celia [15] use generalized functions to facilitate finite element formulation of Laplace's equation. With this unique approach, it becomes clear that the boundary term resulting from application of Green's Theorem is indeed external to the computational domain. The method can also be applied to the coupled hyperbolic system of balance laws considered in this article; an outline of the essential steps in the analysis is given below.

To clarify the presentation, focus on the linearized, steady-state, primitive continuity equation with constant bathymetry. Under these assumptions, equation (1) simplifies to

$$\nabla \bullet \mathbf{v} = 0 \tag{6}$$

Discretize all space, Ω_∞, into triangular elements (element shape is arbitrary). Approximate $\mathbf{v}$ with $\mathbf{v}_e\gamma_e$ (summation implied) where $\mathbf{v}_e$ is a piecewise polynomial approximation of $\mathbf{v}$ inside the domain of interest (Ω), γ_e is a generalized Heaviside step function that has a value of one in element e and zero outside, and the summation is over all elements in Ω_∞. Outside of Ω, the choice of the approximating function for $\mathbf{v}$ is arbitrary; a reasonable choice is to select $\mathbf{v}_e$ so that it satisfies the governing differential equation exactly.

Obtain the finite element approximation by weighting (6) with a set of polynomial weighting functions, φ_i, which are defined to be equal to 1 at node i and zero at all other nodes, and integrating the result over the domain

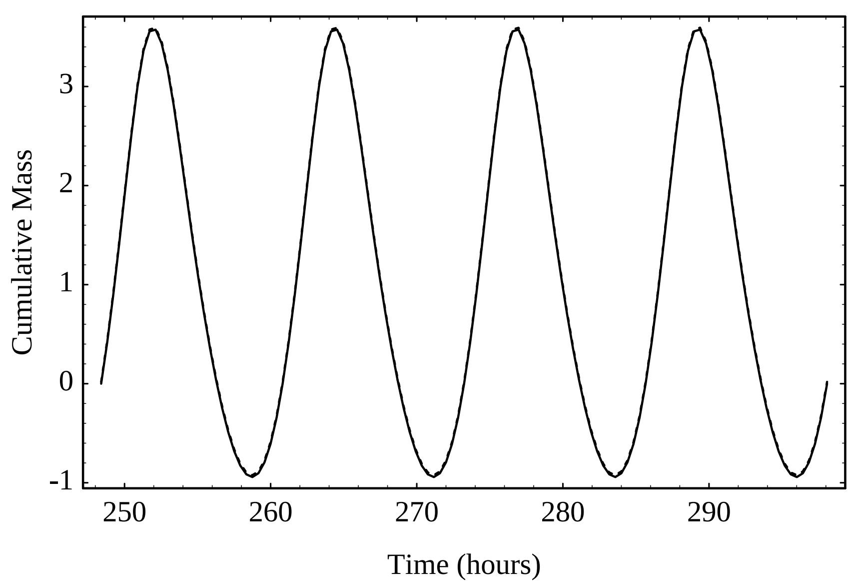

Figure 2 Global mass balance check of the GWC formulation ($G = \tau$) for the model problem, external flux interpretation of boundary conditions.

$$\sum_e \int_{\Omega_\infty} \nabla \bullet (\mathbf{v}_e \gamma_e)\varphi_i d\Omega = 0 \quad \text{for} \quad i = 1, \dots N \tag{7}$$

Next expand the spatial derivative in (7) and use properties of the step function to simplify the resulting terms. These properties are that the sum of γ_e over all elements is one and that the gradient of γ_e is a generalized Dirac function whose integral relations are similar to the familiar one-dimensional Dirac [16]. Thus, integrals involving $\nabla\gamma_e$ are converted to integrals over the boundary of the element where the discontinuity in γ_e occurs. Now, if E is defined to be all elements that have node i in common, then (7) becomes

$$\sum_E \left(\int_{\Omega_E} \nabla \bullet \mathbf{v}_E \varphi_i d\Omega - \int_{\partial\Omega_E} \mathbf{n}_E \bullet \mathbf{v}_E \varphi_i d\Gamma \right) = 0 \quad \text{for} \quad i = 1, \dots N \tag{8}$$

where $\mathbf{n}_E$ is a unit normal pointing out from element E.

Consider the case where i is on the boundary of the domain of interest, Ω. (For the case of i on the interior of the domain, (8) reduces to a familiar finite element approximation.) Define elements A to be those elements of E that are inside Ω and elements B those outside of Ω. Then, it can be shown that (8) simplifies to

$$-\sum_A \int_{\Omega_A} \mathbf{v}_A \bullet \nabla\varphi_i d\Omega + \int_{\partial\Omega} \mathbf{n}_A \bullet \mathbf{v}_B \varphi_i d\Gamma = 0 \tag{9}$$

In the second term, $\mathbf{v}_B$ is external to the domain so its value can be chosen to produce the best solution in Ω. An obvious choice is to select $\mathbf{v}_B$ such that $\mathbf{n}_A \bullet \mathbf{v}_B = 0$ on land boundaries and $\mathbf{n}_A \bullet \mathbf{v}_B$ is equal to the normal flux on open boundaries. Clearly in this formulation, $\mathbf{v}_B$ is interpreted as external to the computational domain.

SUMMARY

Shallow water models based on a finite element solution of the wave continuity equation have evolved as powerful tools for simulating the hydrodynamic behavior of coastal and oceanic waters. Experiments and analyses show that proper interpretation of boundary conditions is needed to improve the accuracy and the mass-conserving properties of the algorithm. In particular, boundary terms that result from the application of Green's Theorem to the weighted residual form of the governing equations should be interpreted as external to the computational domain. This interpretation can be viewed as the finite element counterpart of the use of imaginary nodes (nodes outside the boundary) in finite difference algorithms.

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